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Application of Optimal Homotopy Asymptotic Method for the Approximate Solution of Riccati Equation

(Penggunaan Kaedah Homotopi Asimptotik Optimum untuk Penyelesaian Hampiran Persamaan Riccati)

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ABSTRACT

In this paper, the optimal homotopy asymptotic method (OHAM) is applied to obtain an approximate solution of the nonlinear Riccati differential equation. The method is tested on several types of Riccati differential equations and comparisons that were made with numerical results showed the effectiveness and accuracy of this method.

Keywords: Optimal homotopy asymptotic method; Riccati differential equation

ABSTRAK

Dalam kertas ini, kaedah homotopi asimtotik optimum (OHAM) diaplikasi untuk memperoleh penyelesaian persamaan pembezaan Riccati tak linear. Kaedah ini diuji ke atas beberapa persamaan pembezaan Riccati dan perbandingan keputusan berangka menunjukkan keberkesanan dan kejituan kaedah ini.

Kata kunci: Kaedah homotopi asimptotik optimum; persamaan pembezaan Riccati

INTRODUCTION

The Riccati equation is an important non-linear ordinary differential equation in dynamical systems and is of the form:

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2, \tag{1}$$

where y,P,Q and R are real functions of the real argument x.

This equation has applications in random processes, optimal control and diffusion amongst many other applications. Reid (1972) highlighted some of the fundamental theoretical concepts related to the Riccati equation. Due to its importance, it is essential that the Riccati equation be accurately and efficiently solved.

Approximate analytical methods have been widely used in recent years to solve operator equations. With respect to the Riccati differential equation, El-Tawil et al. (2004) and Wazwaz and Al-Sayed (2001) have used the Adomian decomposition method (ADM), Batiha et al. (2007) have used the variational iteration method (VIM) for solving the different examples of Riccati equation and compared the results with the exact solutions. The use of the homotopy analysis method (HAM) has been explored by Tan and Abbasbandy (2008).

Recently, a new approximate analytical technique called the optimal homotopy asymptotic method (OHAM) has been introduced. OHAM has a built in convergence criteria similar to HAM but has the advantage of being more flexible. The papers of Esmaeilpour and Ganji (2010); Ghoreishi et al. (2011); Herisanu and Marinca (2010); Idrees et al. (2012); Iqbal and Javed (2011); Marinca et al. (2009) and Marinca and Herisanu (2008) have demonstrated the effectiveness and generalizability of OHAM.

In this paper, we reviewed the concept of OHAM and applied it to obtain a reliable approximate solution to the Riccati nonlinear differential equation. We will apply OHAM to some examples of the Riccati equation.

BASIC PRINCIPLES OF OHAM

We review the basic principles of OHAM as expounded by Ghoreishi et al. (2011); Idrees et al. (2012) and Marinca and Herisanu (2008).

Consider the following differential equation and boundary condition:

$$L(u(x)) + g(x) + N(u(x)) = 0,\ B\left(u, \frac{du}{dx}\right) = 0, \tag{2}$$

where L is a linear operator, x denotes independent variable, $u(x)$ is an unknown function, $g(x)$ is a known function, N is a nonlinear operator and B is a boundary operator. An equation known as a deformation equation is constructed:

$$(1-p)[L(\phi(x,p) + g(x)] = \begin{cases} H(p)\left[L\left(\phi(x,p)+g(x)+N\left(\phi(x,p)\right)\right)\right] \\ B\left(\phi(x,p), \dfrac{\partial \phi(x,p)}{\partial x}\right) = 0, \end{cases} \tag{3}$$

where $p \in [0,1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$ and $H(0) = 0$, $\phi(x, p)$ is an unknown function. For $p = 0$ and $p = 1$ we have, $\phi(x,0) = u_0(x)$ and $\phi(x,1) = u(x)$, respectively.

Hence, as p varies from 0 to 1 the solution $\phi(x,p)$ varies from $u_0(x)$ to the solution $u(x)$ where, $u_0(x)$ is obtained from (3) for $p = 0$.

$$L(u_0(x)) + g(x) = 0, \; B\left(u_0, \frac{du_0}{dx}\right) = 0. \quad (4)$$

The auxiliary function *H(p)* is chosen in the form:

$$H(p) = pC_1 + p^2C_2 + \ldots \quad (5)$$

where C_1, C_2, … are constants which are to be determined later.

For solution, $\phi(x, p, C_i)$ is expanded in Taylor's series about p and given:

$$\phi(x, p, C_i) = u_0(x) + \sum_{k=1}^{\infty} u_k(x, C_i)p^k, \; i = 1,2,3,\ldots \quad (6)$$

Substituting (5) and (6) into (3) and equating the coefficients of the like powers of p equal to zero, gives the linear equations as described below:

The zeroth order problem is given by (4) and the first and second order problems are given by the (7) and (8), respectively:

$$L(u_1(x)) = C_1N_0(u_0(x)), \quad B\left(u_1, \frac{du_1}{dx}\right) = 0. \quad (7)$$

$$L(u_2(x)) - L(u_1(x))$$

$$= \begin{cases} C_2N_0(u_0(x)) + C_1[L(u_1(x)) + N_1(u_0(x), u_1(x))] \\ B\left(u_2, \frac{du_2}{dx}\right) = 0. \end{cases} \quad (8)$$

The general governing equations for $u_k(x)$ are given by:

$$L(u_k(x)) - L(u_{k-1}(x)) = C_kN_0(u_0(x))$$

$$+ \sum_{i=1}^{k-1} C_i \, [L(u_{k-1}(x)) + N_{k-1}(u_0(x), u_1(x), \ldots, u_{k-1}(x))]$$

$$B\left(u_k, \frac{du_k}{dx}\right) = 0, \qquad k = 2,3,\ldots, \quad (9)$$

where $N_m(u_0(x), u_1(x), \ldots, u_m(x))$ is the coefficient of p^m in the expansion of about the embedding parameter p.

$$N(\phi(x, p, C_i)) = N_0(u_0(x)) + \sum_{m=1}^{\infty} N_m \, (u_0, u_1, \ldots, u_m)p^m,$$
$$i = 1,2,3\ldots. \quad (10)$$

It has been observed by previous researchers that the convergence of the series (6) is dependent upon the auxiliary constants C_1, C_2, … . If it is convergent at $p = 1$, one has

$$\tilde{u}(x, C_1, C_2, \ldots, C_m) = u_0(x) + \sum_{i=1}^{m} u_i(x, C_1, C_2, \ldots, C_m). \quad (11)$$

Substituting (11) into (2), the general problem, results in the following residual:

$$R(x, C_1, C_2, \ldots, C_m) = L(\tilde{u}(x, C_1, C_2, \ldots, C_m))$$
$$+ N(\tilde{u}(x, C_1, C_2, \ldots, C_m)). \quad (12)$$

If $R = 0$, then $\tilde{u}$ will be the exact solution. For nonlinear problems, generally this will not be the case.

For determining $C_1(i = 1,2,\ldots,m)$, a and b are chosen such that the optimum values for C_i are obtained using the method of least squares:

$$J(C_1, C_2, \ldots, C_m) = \int_a^b R^2(x, C_1, C_2, \ldots, C_m)dx, \quad (13)$$

where $R = L(\tilde{u}) + g(x) + N(\tilde{u})$ is the residual and

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \ldots = \frac{\partial J}{\partial C_m} = 0. \quad (14)$$

With these constants, one can get the approximate solution of order m.

NUMERICAL EXAMPLES

In this section, we solve two examples of Riccati nonlinear differential equations with the use of *Mathematica* software.

Example 1

For $x \in [0,1]$ consider the following nonlinear differential equation,

$$\frac{du}{dx} + u^2 - 1 = 0, \quad u(0) = 0. \quad (15)$$

This problem was considered by Batiha et al. (2007). The exact solution of the problem is given as:

$$u(x) = \frac{e^{2x} - 1}{e^{2x} + 1}. \quad (16)$$

Applying the mentioned method (OHAM), the zeroth, first, second, third, fourth and fifth order problems with initial conditions are as given below, respectively;

$$u_0'(x) = 1, \qquad u_0(0) = 0. \quad (17)$$

$$u_1'(x, C_1) = -1 - C_1 + C_1u_0^2 + (1 + C_1)u_0'(x), \quad u_1(0) = 0. \quad (18)$$

$$u_2'(x, C_1) = 2C_1u_0u_1 + (1 + C_1)u_1'(x), \quad u_2(0) = 0. \quad (19)$$

$$u_3'(x, C_1) = C_1u_1^2 + 2C_1u_0u_1 + (1 + C_1)u_2'(x), \quad u_3(0) = 0. \quad (20)$$

$$u_4'(x, C_1) = 2C_1u_1u_2 + 2C_1u_0u_3 + (1 + C_1)u_3'(x), \quad u_4(0) = 0. \quad (21)$$

$$u_5'(x,C_1) = C_1 u_2^2 + 2C_1 u_1 u_3 + 2C_1 u_0 u_4 + 2u_1 + (1+C_1)u_4'(x), \quad u_5(0) = 0. \tag{22}$$

Solving (17) - (22), we get the fifth-order approximate solution for $p = 1$ as:

$$\tilde{u}(x,C_1) = u_0(x)+u_1(x,C_1)+u_2(x,C_1) +u_3(x,C_1)+u_4(x,C_1)+u_5(x,C_1). \tag{23}$$

We use the method of least squares to obtain the unknown convergent constant in $\tilde{u}$.

$$C_1 = -0.773662564.$$

By considering the value of C_1 and after simplification of (23), the approximate solution becomes,

$$\tilde{u} = x - 0.333135x^3 + 0.131901x^5 - 0.0496428x^7 + 0.0149286x^9 - 0.00245669x^{11}.$$

In Table 1, the obtained solutions using OHAM are compared with the exact, low error is remarkable. In Figure 1 the maximum magnitude of the Residual R($\tilde{u}$) is 0.00004 particularly at x =1, which shows the efficiency of the proposed method.

Example 2

For x ∈ [0,1] consider the following nonlinear differential equation

$$\frac{du}{dx} - 2u + u^2 - 1 = 0, \qquad u(0) = 0. \tag{24}$$

This problem was also considered by Batiha et al. (2007).The exact solution is;

$$u(x) = 1+ \sqrt{2}Tanh\left[\sqrt{2}x + \frac{1}{2}Log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]. \tag{25}$$

Applying OHAM, the zeroth, first, second and third order problems with initial conditions are as given below, respectively;

TABLE 1. The comparison of the solution using OHAM, with exact and VIM

x	OHAM (Present method)	Exact	Ab. Error (VIM) (Batiha et al. 2007)	Ab. Error (OHAM)
0	0	0	0	0
0.1	0.09966	0.09966	5.0×10^{-11}	1.8×10^{-7}
0.2	0.19737	0.19737	4.3×10^{-9}	1.1×10^{-6}
0.3	0.29131	0.29131	1.5×10^{-7}	2.6×10^{-6}
0.4	0.37995	0.37994	1.9×10^{-6}	3.5×10^{-6}
0.5	0.46212	0.46211	1.3×10^{-5}	2.9×10^{-6}
0.6	0.53705	0.53705	6.6×10^{-5}	1.6×10^{-6}
0.7	0.60436	0.60436	2.4×10^{-4}	8.7×10^{-7}
0.8	0.66403	0.66403	7.3×10^{-4}	9.1×10^{-7}
0.9	0.71629	0.71629	1.9×10^{-3}	1.1×10^{-6}
1.0	0.76159	0.76159	4.4×10^{-3}	1.7×10^{-7}

Ab.Error = $|u_{Exact} - u_{approx}|$

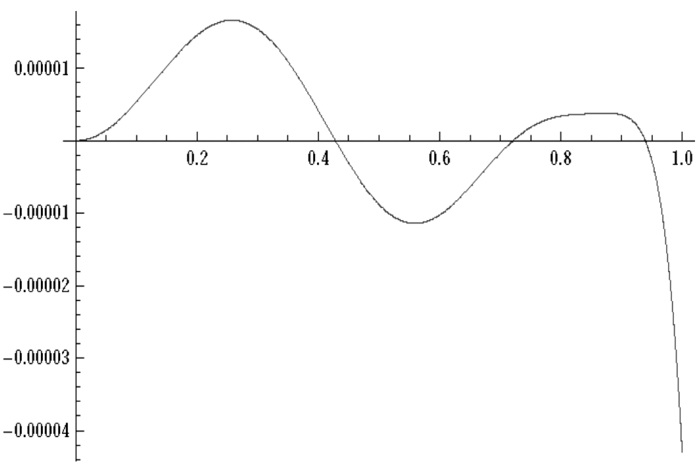

FIGURE 1. The accuracy of OHAM from the plot of residual of Example 1

$$u_0'(x) = 1 + 2u_0, \quad u_0(0) = 0. \tag{26}$$

$$u_1'(x, C_1) = -1 - C_1 - 2u_0 - 2C_1u_0 + C_1u_0^2 + 2u_1 + (1 + C_1)u_0'(x), \quad u_1(0) = 0. \tag{27}$$

$$u_2'(x, C_1, C_2) = -C_2 - 2C_2u_0 + C_2u_0^2 - 2u_1 - 2C_1u_1 + 2C_1u_0u_1 + 2u_2 + C_2u_0' + (1 + C_1)u_1'(x), \quad u_2(0) = 0. \tag{28}$$

$$u_3'(x, C_1, C_2, C_3) = -C_3 - 2C_3u_0 + C_3u_0^2 - 2C_2u_1 + 2C_2u_0u_1 + C_1u_1^2 - 2u_2 - 2C_1u_2 + 2C_1u_0u_2 + 2u_3 + C_3u_0' + C_2u_1' + (1 + C_1)u_2'(x), \quad u_3(0) = 0. \tag{29}$$

Solving (26) - (29), to get the third-order approximate solution for $p = 1$ as:

$$\tilde{u}(x,C_1,C_2,C_3) = u_0(x) + u_1(x,C_1) + u_2(x,C_1,C_2) + u_3(x,C_1,C_2,C_3). \tag{30}$$

The method of least squares was used to obtain the unknown convergent constants in $\tilde{u}$.

C_1 = -0.586894206291958, C_2 = 0.013204624382111, C_3 = -0.000341941888771.

By putting the values of C_1, C_2, C_3 and after simplification of (30), we get the approximate solution,

$$\tilde{u} = x + x^2 + 0.360356x^3 - 0.279289x^4 - 0.458747x^5 - 0.254717x^6 + 0.0198107x^7 + 0.0162312x^7 + 0.158434x^9 + 0.0856419x^{10} + 0.0146111x^{11} - 0.0254594x^{12} - 0.036575x^{13} - 0.0316362x^{14} - 0.0218031x^{15},$$

In Table 2, we observe the same behavior as in Table 1. From Figure 2 the maximum magnitude of the Residual R ($\tilde{u}$) is 0.05 particularly at $x = 1$ which shows the accuracy of method.

TABLE 2. The comparison of the solution using OHAM, with exact and VIM

x	OHAM (Present method)	Exact	Ab. Error (VIM) (Batiha et al. 2007)	Ab. Error (OHAM)
0	0	0	0	0
0.1	0.11032	0.11029	1.9×10^{-8}	3.2×10^{-5}
0.2	0.24227	0.24197	1.0×10^{-6}	2.9×10^{-4}
0.3	0.39618	0.39510	8.8×10^{-6}	1.1×10^{-3}
0.4	0.57036	0.56781	3.3×10^{-5}	2.5×10^{-3}
0.5	0.76044	0.75601	7.2×10^{-5}	4.4×10^{-3}
0.6	0.95939	0.95356	9.9×10^{-5}	5.5×10^{-3}
0.7	1.15854	1.15295	8.8×10^{-5}	5.5×10^{-3}
0.8	1.35019	1.34636	1.5×10^{-5}	3.8×10^{-3}
0.9	1.53016	1.52691	4.9×10^{-4}	3.2×10^{-3}
1.0	1.69294	1.68951	3.4×10^{-3}	3.4×10^{-3}

$Ab.Error = |u_{Exact} - u_{approx}|$

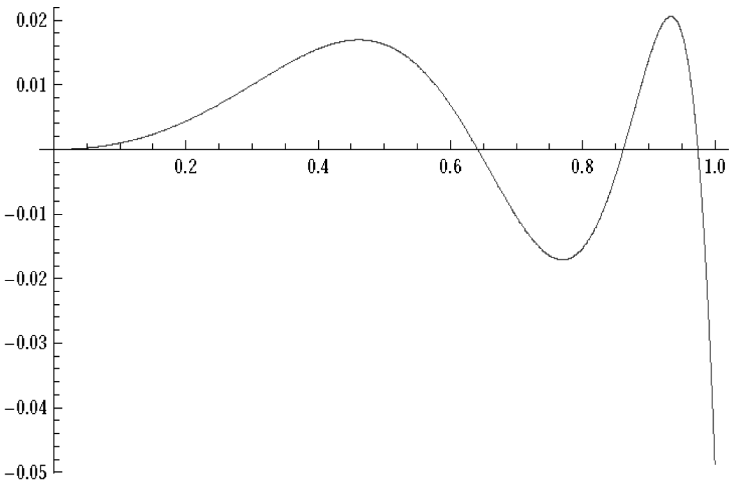

FIGURE 2. The accuracy of OHAM from the plot of residual of Example 2

Conclusion

In this paper, we studied the approximate analytical solution of nonlinear Riccati differential equation. We have applied a recently introduced technique called the optimal homotopy asymptotic method to solve this nonlinear differential equation using the *Mathematica* software. The obtained results suggest that the OHAM could be a useful and effective tool in solving nonlinear differential equations. The procedure has advantages over some existing analytical approximation methods. The convergence and low error for OHAM is remarkable.

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